MODULE – IV

UNSYMMETRICAL BENDING

BENDING OF CURVED BEAMS

INTRODUCTION TO ENERGY METHODS

UNSYMMETRICAL BENDING

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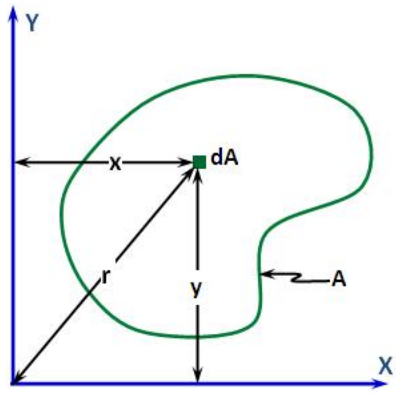
MOMENT OF INERTIA OF AN AREA:

$$I_{xx} = \int_A y^2 dA$$

$$I_{yy} = \int_A x^2 dA$$

The first two integrals are known as moment of inertia of area about x and y axis respectively.



****They are called so because of the similarity with integrals that define the mass moment of inertia of bodies in the field of dynamics. Since an area cannot have an inertia, the terminology moment of inertia of an area is a misnomer. This terminology for the above integral has become a common usage.***

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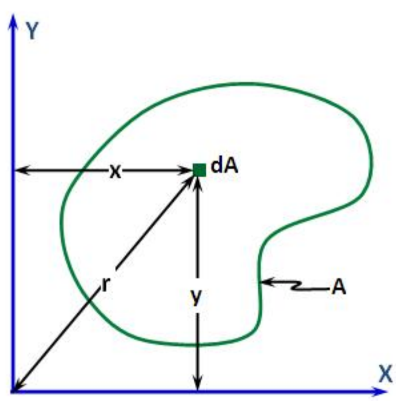
MOMENT OF INERTIA OF AN AREA:

$$I_{xy} = \int_A xy dA$$

The above integral is called product of inertia. Its sign can be positive or negative.

$$J = \int_A r^2 dA$$

The above integral is called polar moment of inertia of the area.



It is the moment of an area about an axis perpendicular to the x and y axis. Polar moment of inertia of an area is the sum of moment of inertia about x and y axis.

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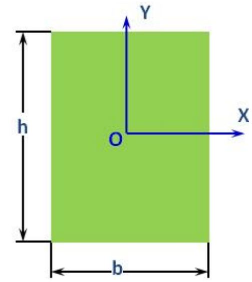
Moment of Inertia of some common Area:

1. **Rectangle:**

$$I_{xx} = \frac{bh^3}{12} \qquad J_o = \frac{bh^3 + hb^3}{12}$$

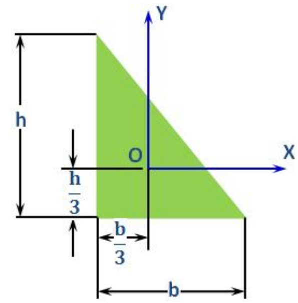
$$I_{yy} = \frac{hb^3}{12} \qquad I_{xy} = 0$$



2. **Right Triangle:**

$$I_{xx} = \frac{bh^3}{36} \qquad J_o = \frac{bh^3 + hb^3}{36}$$

$$I_{yy} = \frac{hb^3}{36} \qquad I_{xy} = \frac{b^2h^2}{72}$$



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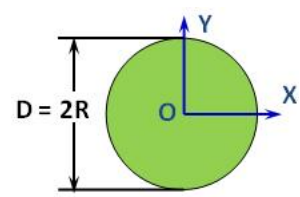
3. **Circle:**

$$I_{xx} = \frac{\pi D^4}{64} = \frac{\pi R^4}{4} \qquad J_o = \frac{\pi D^4}{32} = \frac{\pi R^4}{2}$$

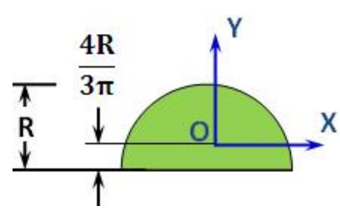
$$I_{yy} = \frac{\pi D^4}{64} = \frac{\pi R^4}{4} \qquad I_{xy} = 0$$



3. **Semicircle:**

$$I_{xx} = \pi R^4 \left(\frac{1}{8} - \frac{8}{9\pi^2}\right) \qquad I_{xy} = 0$$

$$I_{yy} = \frac{\pi R^4}{8} \qquad J_o = \pi R^4 \left(\frac{1}{4} - \frac{8}{9\pi^2}\right)$$



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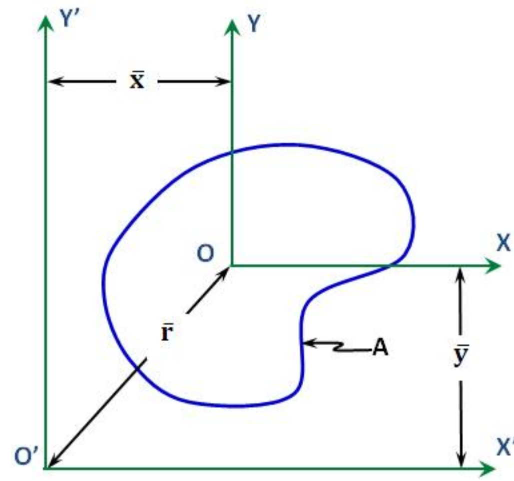
PARALLEL AXIS THEOREM:

$$I_{x'x'} = I_{xx} + A\bar{y}^2$$

$$I_{y'y'} = I_{yy} + A\bar{x}^2$$

$$J_{o'} = J_o + A\bar{r}^2$$

$$I_{x'y'} = I_{yx} + A\bar{x}\bar{y}$$



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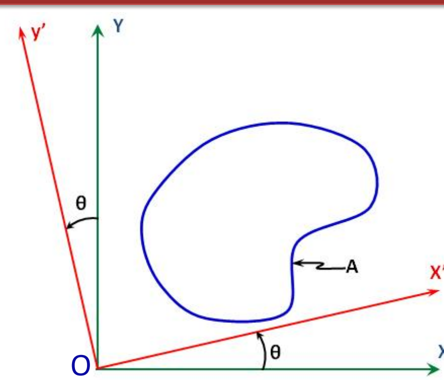
TRANSFORMATION EQUATIONS:

The moments of inertia given with respect to a given set of coordinates xy can be transformed to a new set of coordintes x'y' which makes an angle θ with respect to original set of co ordinates xy can be done using the following transformation equations



$$I_{x'x'} = I_{xx}Cos^2\theta + I_{yy}Sin^2\theta - I_{xy}Sin2\theta$$

$$I_{y'y'} = I_{xx}Sin^2\theta + I_{yy}Cos^2\theta + I_{xy}Sin2\theta$$

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2} Sin\, 2\theta + I_{xy}Cos2\theta$$

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Note the similarity between the transformation equations for moments and products of inertia and the transformation equations of stress.

There are two values of θ for which $I_{xy} = 0$.

These two axes X1X1 & Y1Y1 for which $I_{xy} = 0$ are called **principal axes.**

An axis of symmetry will always be a principal axis.

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$$\tan 2\theta = \frac{2I_{xy}}{I_{XX} - I_{YY}}$$

$$I_{X1X1} = \frac{I_{XX} + I_{YY}}{2} + \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + I_{XY}^2}$$

$$I_{Y1Y1} = \frac{I_{XX} + I_{YY}}{2} - \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + I_{XY}^2}$$

$\mathbf{I_{X1X1}}$ represents maximum moment of inertia and $\mathbf{I_{Y1Y1}}$ represents minimum moment of inertia.

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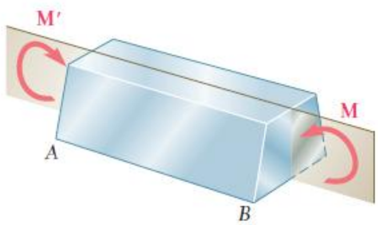
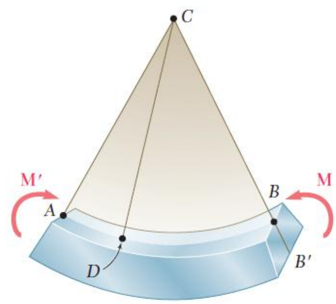
Symmetrical Bending: In the case of symmetrical bending, it is essential that the plane containing one of the principal axis of inertia, the plane of applied moment and the plane of deflection should coincide. The neutral axis will coincide with the other principal axis of inertia.





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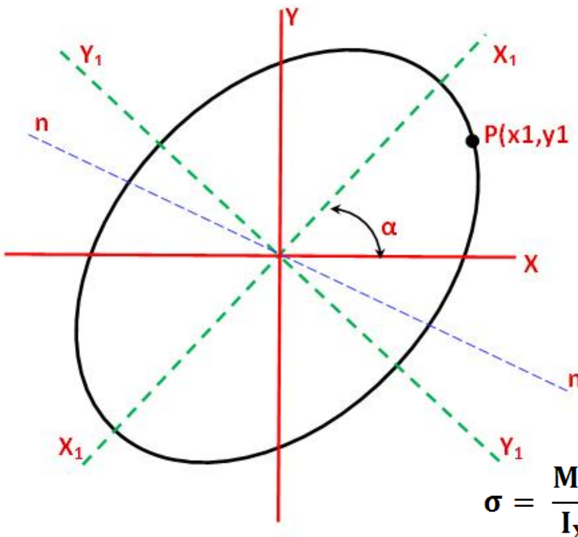
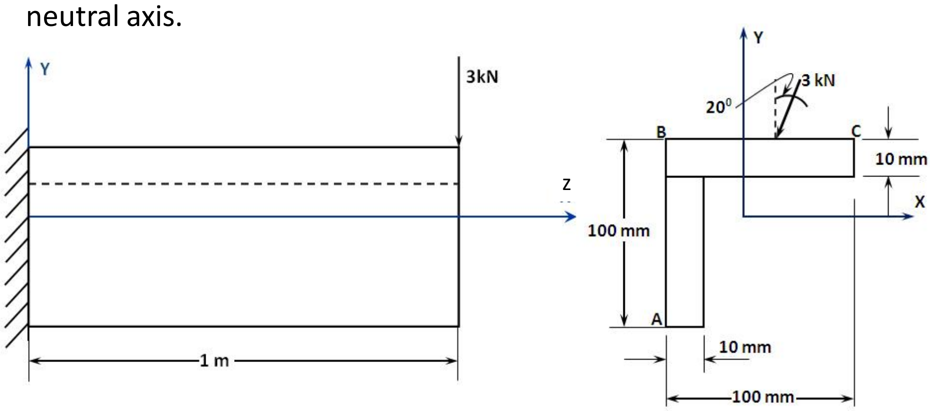


$$\sigma = \frac{M_{X1X1}}{I_{X1X1}} y1 + \frac{M_{Y1Y1}}{I_{Y1Y1}} X1$$

UNSYMMETRICAL BENDING

A cantilever of angle is 1 m long and is fixed at one end, while it is subjected to a load of 3 kN at the free end at 20^0 to the vertical. Calculate the bending stress at A, B and C and also the position of neutral axis.



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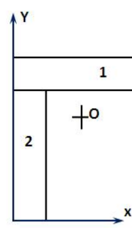
Locating the centroid of the cross section

Sl. No	b (mm)	h (mm)	Area (mm^2)	$\bar{x}$	$\bar{y}$	First Moment about x axis	First Moment about y axis
1	100	10	1000	50	95	95000	50000
2	10	90	900	5	45	40500	4500
		Sum:	1900		Sum:	135500	54500
			$\bar{X}$:	28.7 mm		$\bar{Y}$:	71.31 mm



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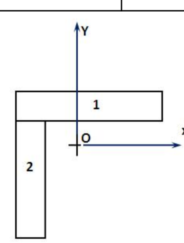
Moment of inertia about the centroidal axis of the cross section:

Sl. No	b	h	Area	$\bar{x}$	$\bar{y}$	I_{xx} about local centroid	I_{yy} about local centroid	I_{XX} about global centroid	I_{YY} about global centroid
1	100	10	1000	21.3	23.7	8333.33	833333.3	570023.3	1287023
2	10	90	900	23.7	-26.3	607500	7500	1230021	513021
			1900					1800044	1800044

$$I_{xx} = \frac{bh^3}{12} \qquad I_{XX} = I_{xx} + A\bar{Y}^2$$

$$I_{yy} = \frac{hb^3}{12} \qquad I_{YY} = I_{yy} + A\bar{X}^2$$



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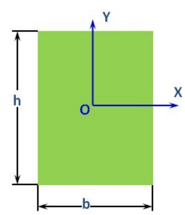
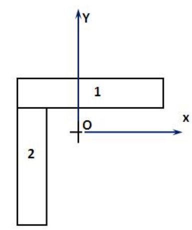
Product of inertia (I_{XY}) about the centroidal axis of the cross section:

Sl. No	b	h	Area	$\bar{x}$	$\bar{y}$	I_{XY} about global centroid (mm^4)
1	100	10	1000	21.3	23.7	504810
2	10	90	900	-23.7	-26.3	560979
			1900			1065789

$$I_{xy} = 0$$

$$I_{XY} = I_{xy} + A\bar{x}\,\bar{y}$$





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Finding Principal Axis:

$$\tan 2\theta = \frac{2I_{xy}}{I_{XX} - I_{YY}}$$

$$I_{X1X1} = \frac{I_{XX} + I_{YY}}{2} + \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + {I_{XY}}^2}$$

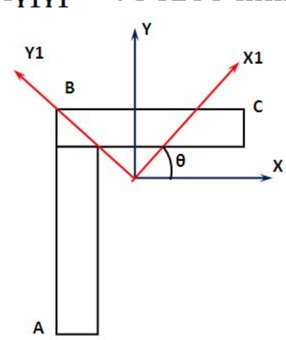
$$I_{Y1Y1} = \frac{I_{XX} + I_{YY}}{2} - \sqrt{\left(\frac{I_{XX} - I_{YY}}{2}\right)^2 + {I_{XY}}^2}$$

$$\tan 2\theta = \infty$$

$$\theta = 45^0, 135^0$$

$$I_{X1X1} = 2865833\ mm^4$$

$$I_{Y1Y1} = 734255\ mm^4$$



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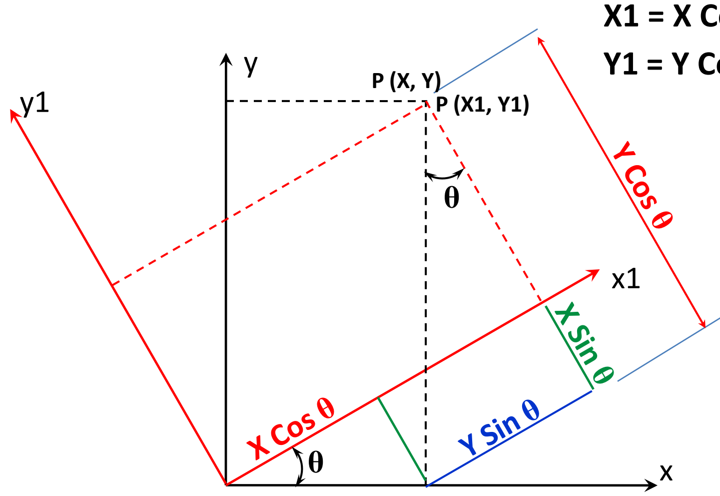
Coordinates of a point With reference to Principal Axis

X1 = X Cosθ + Y Sinθ

Y1 = Y Cosθ – X Sinθ



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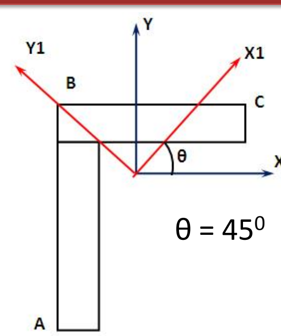
Coordinates of A B & C With reference to Principal Axis

X1 = X Cosθ + Y Sinθ

Y1 = Y Cosθ – X Sinθ



$\theta = 45^0$

Point	X	Y	X1	Y1
A	-28.7	-71.31	-70.71	-30.13
B	-28.7	28.7	0	40.59
C	71.31	28.7	70.71	-30.13

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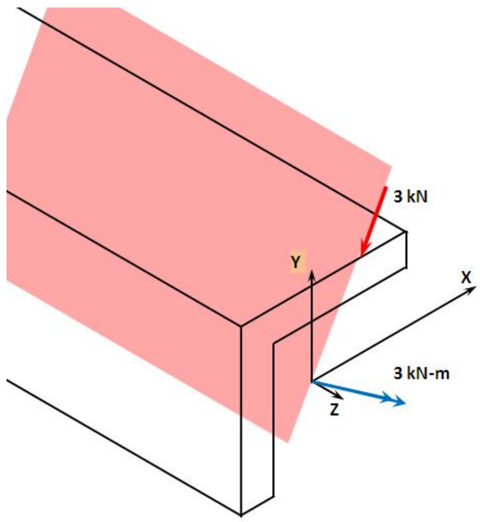
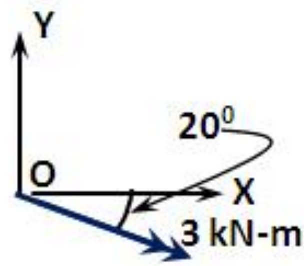
Components of Moments along X1Z plane and Y1Z plane:





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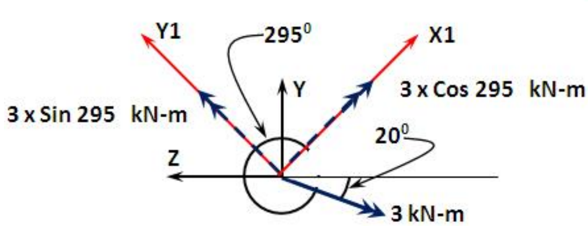
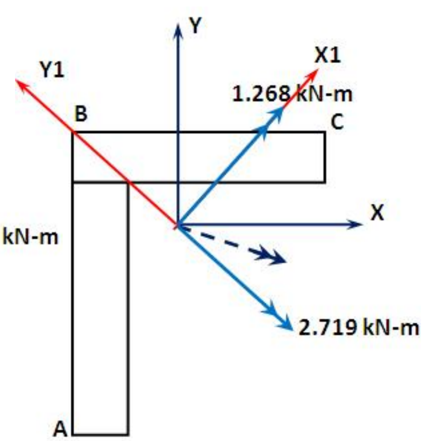
Components of Moments along X1Z plane and Y1Z plane:





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Stresses due to bending:

$$\sigma = \frac{M_{X1X1}}{I_{X1X1}} y1 + \frac{M_{Y1Y1}}{I_{Y1Y1}} X1$$

M_{X1} (N-mm): 1268000 **I_{X1X1} (mm^4):** 2865833

M_{Y1} (N-mm): 2719000 **I_{Y1Y1} (mm^4):** 734255

Point	Y1	X1	Stress (MPa)
A	**-30.13**	-70.71	-275.18
B	**40.59**	0	17.96
C	**-30.13**	70.71	248.51

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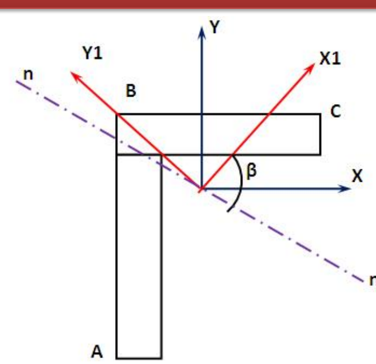
Neutral Axis:



$$\frac{M_{X1}y1}{I_{X1X1}} + \frac{M_{y1}X1}{I_{y1y1}} = 0$$

$$\tan\beta = \frac{y1}{x1} = -\frac{M_{Y1}I_{X1X1}}{M_{X1}I_{y1y1}}$$

$$\tan\beta = \frac{y1}{z1} = -\frac{-2719000 \times 2865833}{1268000 \times 734255}$$

$$= -8.37$$

$\beta = -83.2^0$

UNSYMMETRICAL BENDING

Summary:
Locate the Centroid of the cross section.
Draw x & y axis throug

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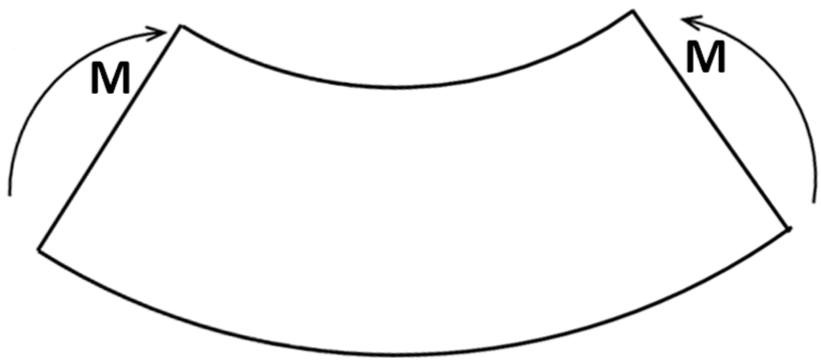
WINCKLER BACH FORMULA

BENDING OF CURVED BEAMS



Consider a curved beam subjected to bending moment M.

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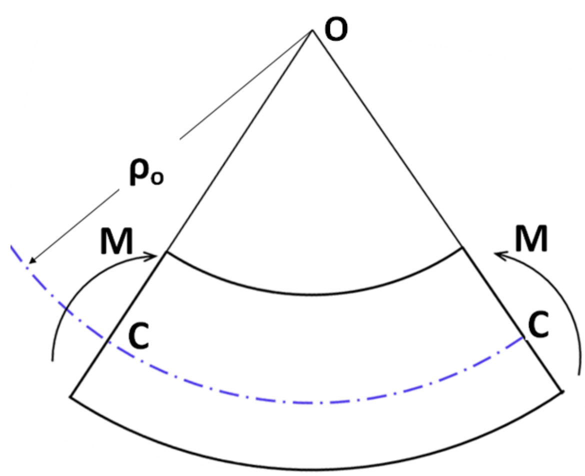
O is the initial centre of curvature of the beam.

CC is the trace of the layer through the centroid.

ρ_0 is the radius of curvature of the centroidal axis CC

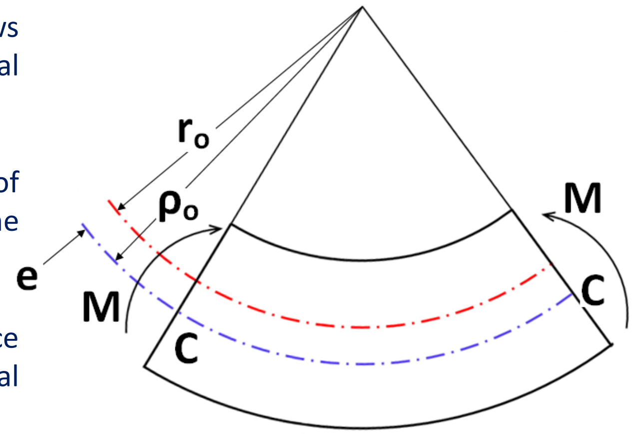


BENDING OF CURVED BEAMS

Red chain line shows the trace of neutral layer.

r_0 is the radius of curvature of the neutral axis.

e is the radial distance between the centroidal axis and neutral axis.



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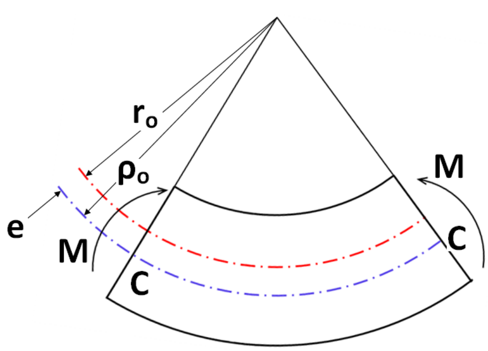

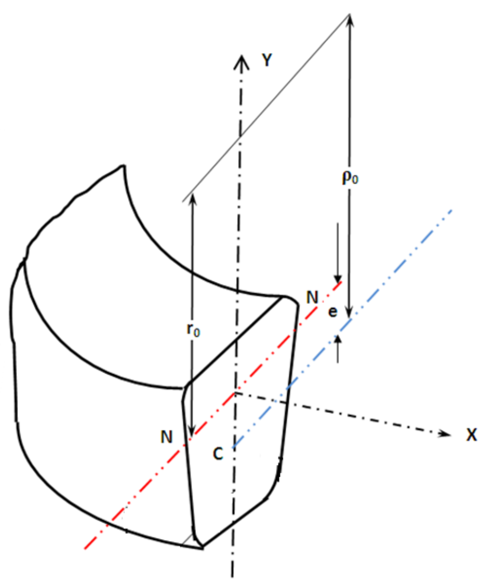

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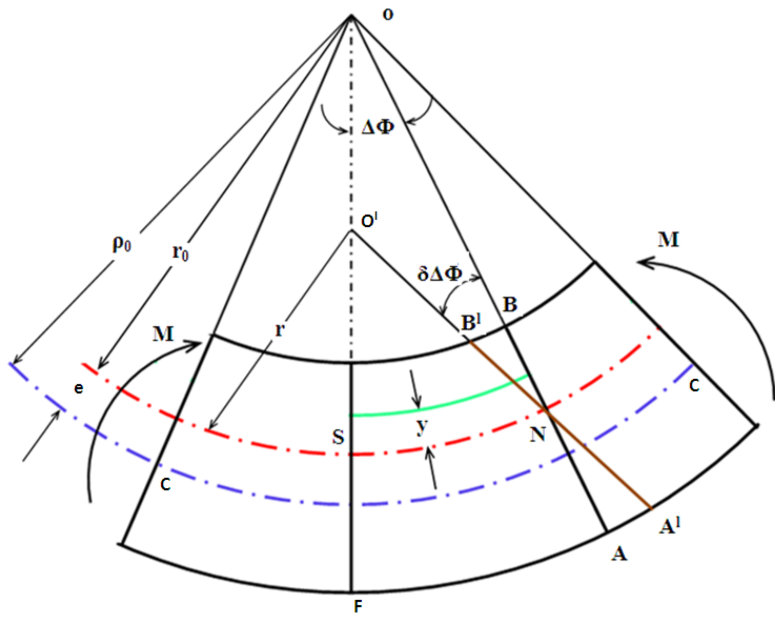

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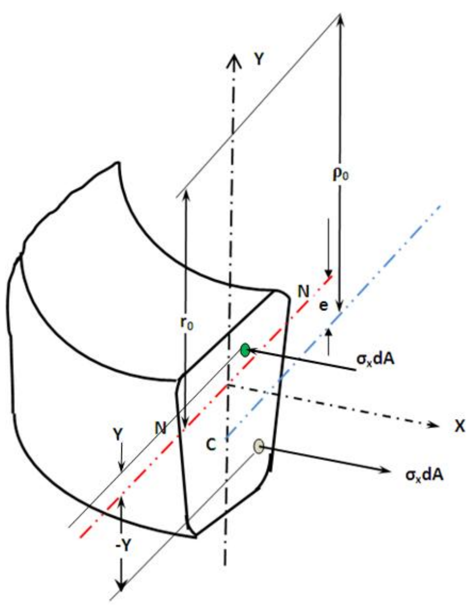

BENDING OF CURVED BEAMS

Consider the bending of a beam which is initially curved. Let an arbitrary length encloses an angle $\Delta\Phi$. Owing to the moment M, the section AB rotates through an angle $\delta\Delta\Phi$ and occupies a position A'B'

ρ_0 – initial radius of curvature.

r_0 – radius of curvature of the neutral surface.

r – final radius of curvature of neutral surface.

SN – Trace of neutral layer.

Assume that the sections which are plane before bending remains plane after bending.

BENDING OF CURVED BEAMS

Hence a transverse section rotates about the neutral axis.

The section AB rotates about the neutral axis NN. Fibers above the neutral layer gets compressed and fibers below the neutral layer gets stretched. The length of fibers in the neutral layer remains unaltered.

Consider a fiber at a distance y from the neutral surface. The unstretched length before bending is $(r_0-y)\Delta\Phi$.

Change in length due to bending is $y\delta\Delta\Phi$.

BENDING OF CURVED BEAMS

The strain is negative for positive y, for the moment shown in figure,

Strain $$\varepsilon_x = -\frac{y\delta\Delta\Phi}{(r_0 - y)\Delta\Phi} \quad \text{— } 1$$

The quantity y remains unaltered during bending. From the figure,

$$SN = (\Delta\Phi + \delta\Delta\Phi)r$$

Also from the figure $SN = r_0\Delta\Phi$

This implies, $(\Delta\Phi + \delta\Delta\Phi)r = r_0\Delta\Phi$

$$\frac{\delta\Delta\Phi}{\Delta\Phi} = \frac{r_0}{r} - 1 \qquad \frac{\delta\Delta\Phi}{\Delta\Phi} = r_0\left(\frac{1}{r} - \frac{1}{r_0}\right) \quad \text{— } 2$$

BENDING OF CURVED BEAMS

Substituting in eqn. 1 we get,

$$\varepsilon_x = -\frac{y}{(r_0 - y)} r_0\left(\frac{1}{r} - \frac{1}{r_0}\right) \quad \text{— } 3$$

Assuming that only σ_x exist,

$$\sigma_x = E\varepsilon_x$$

$$\sigma_x = -\frac{Ey}{(r_0 - y)} r_0\left(\frac{1}{r} - \frac{1}{r_0}\right) \quad \text{— } 4$$

For equilibrium, the resultant of σ_x the area has to be zero. The resultant moment of σ_x about NN is equal to the applied bending moment.

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$$\int_A \sigma_x dA = 0$$

$$-E\, r_0 \left(\frac{1}{r} - \frac{1}{r_0}\right) \int_A \frac{y dA}{(r_0 - y)} = 0$$

$$\int_A \frac{y dA}{(r_0 - y)} = 0$$

$$-\int_A \sigma_x y dA = M$$

$$E\, r_0 \left(\frac{1}{r} - \frac{1}{r_0}\right) \int_A \frac{y^2 dA}{(r_0 - y)} = M$$

$$E\, r_0 \left(\frac{1}{r} - \frac{1}{r_0}\right) \left[- \int_A y dA + r_0 \int_A \frac{y dA}{(r_0 - y)}\right] = M$$

BENDING OF CURVED BEAMS

$$\int_A \frac{y^2 dA}{(r_0 - y)} = \int_A \frac{-r_0 y + y^2 + r_0 y}{(r_0 - y)} dA$$

$$= \int_A \frac{-r_0 y + y^2}{(r_0 - y)} dA + \int_A \frac{r_0 y}{(r_0 - y)} dA$$

$$= \int_A \frac{-y(r_0 - y)}{(r_0 - y)} dA + r_0 \int_A \frac{y}{(r_0 - y)} dA$$

$$= \int_A -y\, dA + r_0 \int_A \frac{y}{(r_0 - y)} dA$$

BENDING OF CURVED BEAMS

In the above equation, the first integral is the moment of the section w.r.t. the neutral axis and is equal to –Ae.

Where, A is the cross sectional area and e is the distance to the centroid from the neutral axis and this moment is negative.

The second integral is zero.

$$Er_0\left(\frac{1}{r}-\frac{1}{r_0}\right)Ae = M$$

Substituting from eq. 4 we get

BENDING OF CURVED BEAMS

$$-\frac{\sigma_x(r_0-y)}{y}A.e = M$$

$$\sigma_x = \frac{-M}{A.e}\cdot\frac{y}{r_0-y}$$

$$\int_A \frac{ydA}{(r_0-y)} = 0 \qquad u = r_0 - y$$

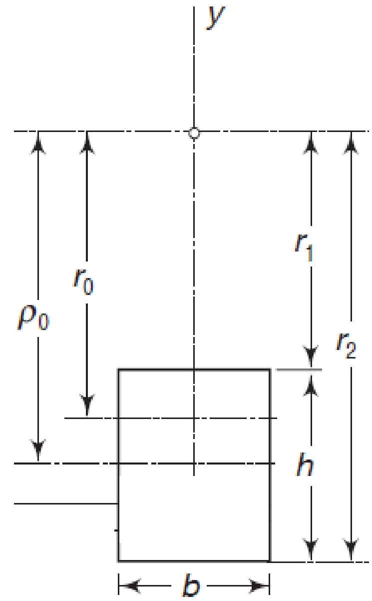
$$\int_A \frac{r_0-u}{u}dA = 0 \Rightarrow \int_A \frac{r_0}{u}dA = \int_A dA$$

$$\Rightarrow r_0\int_A \frac{1}{u}dA = A \Rightarrow r_0 = \frac{A}{\int_A dA/U}$$

BENDING OF CURVED BEAMS

RECTANGULAR SECTIONS

$$r_0 = \frac{h}{Ln\left({}^{r_2}/_{r_1}\right)}$$



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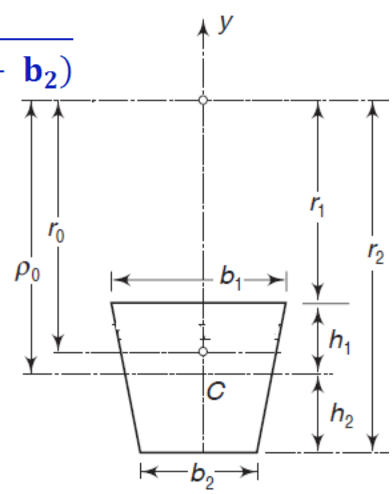
TRAPEZOIDAL SECTIONS

$$\mathbf{r_0} = \frac{\frac{1}{2}\,\mathbf{h^2(b_1 + b_2)}}{\mathbf{(b_1 r_2 - \ b_2 r_1) ln\frac{r_2}{r_1} - \ h(b_1 - \ b_2)}}$$



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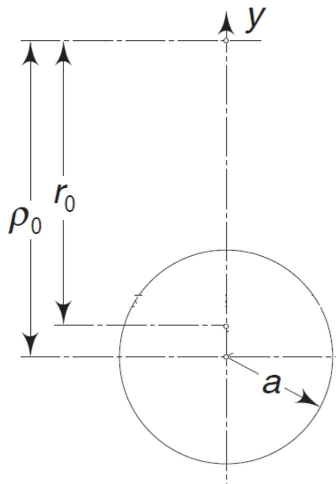
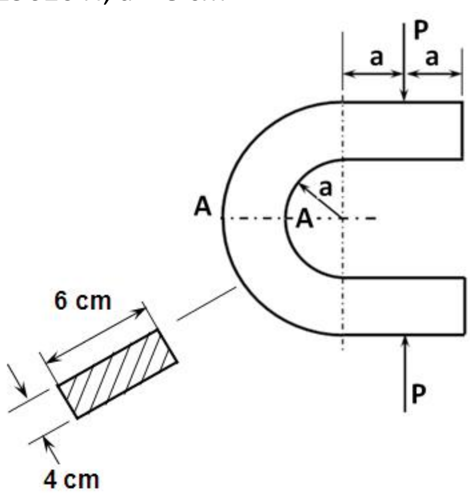
CIRCULAR SECTION



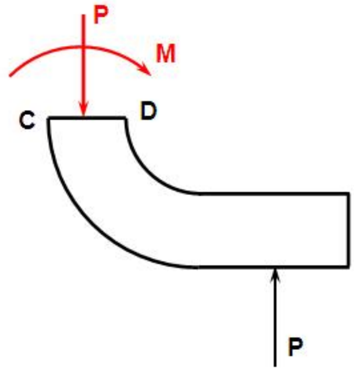
$$r_0 = \frac{1}{2}\left(\rho_0 + \sqrt{\rho_0^2 - a^2}\right)$$

BENDING OF CURVED BEAMS

Determine the maximum tensile stress and maximum compressive stress across section AA of the member shown in fig. Load P = 19620 N, a = 8 cm



BENDING OF CURVED BEAMS



BENDING OF CURVED BEAMS

$$\mathbf{M = 19620\ x\ (30 + 80 + 80) = \ 3727800\ N - mm}$$

$$\boldsymbol{\rho_0 = 80 + 30 = 110\ mm}$$

$$\mathbf{r_0 = \frac{h}{Ln(^{r_2}/_{r_1})}} \qquad \mathbf{r_0 = \frac{60}{Ln\left(^{140}/_{80}\right)} = 107.22\ mm}$$

$$\mathbf{e = \ \rho_0 - \ r_0 = 110 - 107.22 = 2.78\ mm}$$

$$\mathbf{Area, A = \ 60\ x\ 40 = 2400\ mm^2}$$

BENDING OF CURVED BEAMS

At C,

Stress due to bending is given by $(\sigma_x{}^{|})_C = -\frac{M}{A.e} \cdot \frac{y}{(r_0 - y)}$

Stress due to load P is given by $(\sigma_x{}^{||})_C = \frac{P}{Area}$

At C, y = -32.78 mm,

$$(\sigma_x{}^{|})_C = -\frac{3727800}{2400 \text{ x } 2.78} \cdot \frac{-32.78}{(107.22 + 32.78)} \quad (\sigma_x{}^{|})_C = 130.82 \text{ MPa}$$

$$(\sigma_x{}^{||})_C = \frac{19620}{2400} \qquad (\sigma_x{}^{||})_C = -8.175 \text{ MPa}$$

$$(\sigma_x)_C = (\sigma_x{}^{|})_C + (\sigma_x{}^{|})_C = 122.65 \text{ MP}$$

BENDING OF CURVED BEAMS

At D,

Stress due to bending is given by $(\sigma_x{}^{|})_D = -\frac{M}{A.e} \cdot \frac{y}{(r_0 - y)}$

Stress due to load P is given by $(\sigma_x{}^{||})_D = \frac{P}{Area}$

At D, y = 27.22 mm,

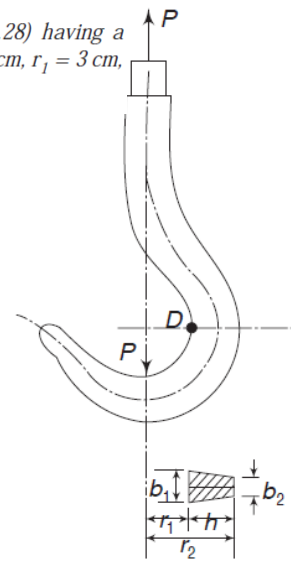
$$(\sigma_x{}^{|})_D = -\frac{3727800}{2400 \text{ x } 2.78} \cdot \frac{27.22}{(107.22 + 27.22)} \quad (\sigma_x{}^{|})_D = -190.11 \text{ MPa}$$

$$(\sigma_x{}^{||})_D = \frac{19620}{2400} \qquad (\sigma_x{}^{||})_D = -8.175 \text{ MPa}$$

$$(\sigma_x)_D = (\sigma_x{}^{|})_D + (\sigma_x{}^{|})_D = 198.3 \text{ MPa}$$

BENDING OF CURVED BEAMS

Determine the stress at point D of a hook (Fig. 6.28) having a trapezoidal section with the following dimensions: $b_1 = 4$ cm, $b_2 = 1$ cm, $r_1 = 3$ cm, $r_2 = 10$ cm, $h = 7$ cm, force $P = 29400$ N.



BENDING OF CURVED BEAMS

$A = 17.5\ \text{cm}^2$

$M = 19P$

$r_0 = 5.204$ cm

$(\sigma'_x)_D = 120148$ kPa

$$x = \frac{(b_1 + 2b_2)\,h}{3(b_1 + b_2)}$$

$(\sigma_x)_D = 136907$ k Pa

$\rho_0 = 5.80$ cm

$e = 0.596$ cm

STRAIN ENERGY OF DEFORMATION:

STRAIN ENERGY OF DEFORMATION

STRAIN ENERGY OF DEFORMATION:

1. Strain energy of deformation -
   a) special cases of a body subjected to concentrated loads
   b) due to axial force
   c) shear force
   d) bending moment
   e) Torque
2. Reciprocal relation -Maxwell reciprocal theorem

STRAIN ENERGY OF DEFORMATION:

Energy Methods are widely used for solving Elastic Problems.

Energy is a scalar quantity, so energy methods are also called scalar methods / Lagrangian Mechanics.

Legrangian Mechanics – principle of conservation of energy.

Newtonian Mechanics - static equilibrium equations.

Energy methods are useful in solving indeterminate problems, members subjected to impact loads, instability problems.

STRAIN ENERGY OF DEFORMATION:

Strain Energy:

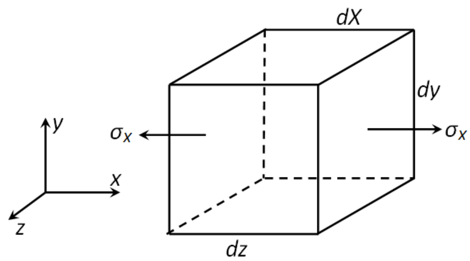
When a body undergoes deformation under the action of externally applied forces, the work done by these forces s stored as strain energy inside the body which can be recovered when the later is elastic in nature.

Strain energy of a member is defined as the increase in energy associated with the deformation of a member. The strain energy density of a material is expressed as strain energy per unit volume.

Strain energy density is equal to the area under the stress-strain diagram.

STRAIN ENERGY OF DEFORMATION:

Consider an infinitesimal element subjected to normal stress.



The force acting on the two faces = σ_x.dy.dz.

Elongation of the element due to this force = ε_x.dx.

The average force during the elongation = (σ_x.dy.dz)/2.

STRAIN ENERGY OF DEFORMATION:

This average force multiplied by the distance through which it acts is the work done on the element.

For a perfectly elastic body no energy is dissipated and the work done on the element is stored as recoverable internal strain energy.

Strain energy U for an infinitesimal element subjected to uniaxial stress is

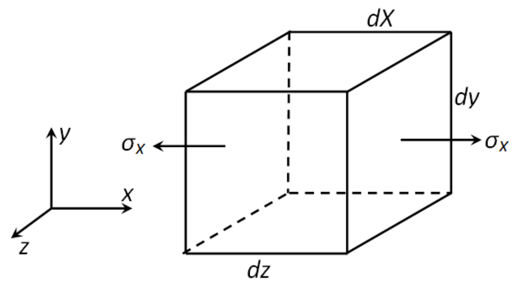
$$\mathbf{du} = \underbrace{\frac{1}{2}\,.\,\sigma_x.\,\mathbf{dy.dz}}_{\text{Average Force}}\ \mathbf{x}\ \underbrace{\varepsilon_x.\,dx}_{\text{distance}} = \frac{1}{2}\,.\,\sigma_x.\ \varepsilon_x.\,\mathbf{dV}$$

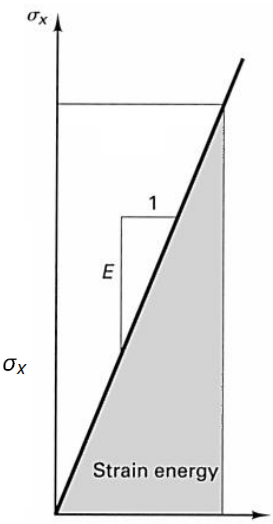
Where dv is the volume of the element.

STRAIN ENERGY OF DEFORMATION:

Strain energy stored per unit volume or strain energy density is given by

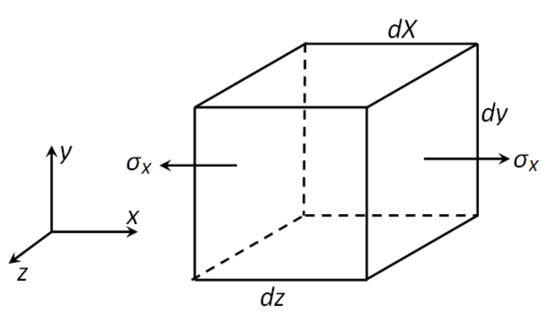
$$U_0 = \frac{1}{2} \cdot \sigma_x \cdot \varepsilon_x$$

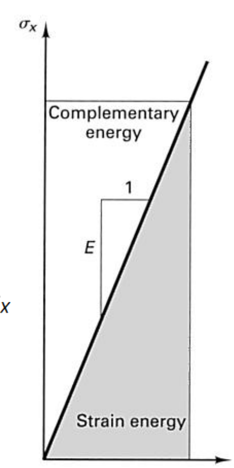




STRAIN ENERGY OF DEFORMATION:

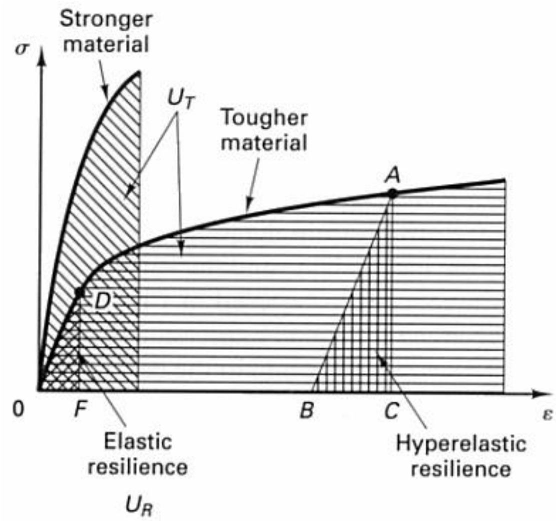
Complimentary Strain Energy:





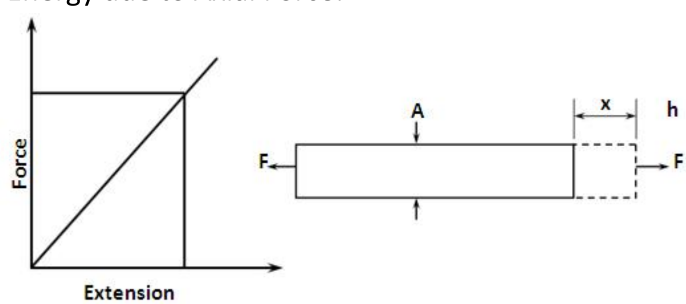
STRAIN ENERGY OF DEFORMATION:

Modulus of resilience and Toughness:



STRAIN ENERGY OF DEFORMATION:

1. Strain Energy due to Axial Force:



Consider a bar of length ‘L’ and cross sectional area ‘A’. The bar is stretched when tensile forces are applied. The graph of Force versus extension is usually a straight line as shown in fig. When the force reaches a value ‘F’ the corresponding extension be ‘x’

STRAIN ENERGY OF DEFORMATION:

Work done by the force $= \frac{1}{2}\mathbf{Fx}$

Strain energy stored, U $= \frac{1}{2}\mathbf{Fx} \quad = \frac{1}{2}\boldsymbol{\sigma\varepsilon}\mathbf{AL}$

(because $\boldsymbol{\sigma} = \frac{F}{A}$ **and** $\boldsymbol{\varepsilon} = \frac{x}{L}$)

$$\mathbf{U} = \frac{1}{2}\boldsymbol{\sigma\varepsilon}\ \mathbf{x\ Volume}$$

In general, strain energy, $\mathbf{U} = \int_V \frac{1}{2}\boldsymbol{\sigma\varepsilon}\mathbf{dV}$

Within the proportionality limit, $\boldsymbol{\varepsilon} = \frac{\boldsymbol{\sigma}}{\mathbf{E}}$

Where E is the Young's modulus

STRAIN ENERGY OF DEFORMATION:

In general, $\mathbf{U} = \int_V \frac{\boldsymbol{\sigma}^2}{2\mathbf{E}}\mathbf{dV}$

The expression for the strain energy in a three dimensional state of stress is given by

$$\mathbf{U} = \frac{1}{2}\int_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})\mathbf{dV}$$

STRAIN ENERGY OF DEFORMATION:

A steel rod has a square section of 10 mm x 10mm and a length of 2 m. Calculate the strain energy when a stress of 400 MPa is produced by stretching it. (Take E = 200 GPa.)

Area A = 10 x 10 = 100 mm^2

= 10^{-4} m^2

Length L = 2 m

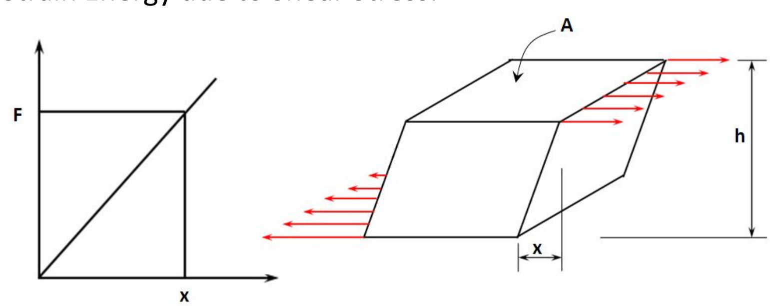
σ = 400 x 10^6 Pa.

E = 200 x 10^9 Pa.

$$U = \frac{\sigma^2}{2E} \text{ x Volume } = \frac{400 \text{ x } 400 \text{ x } 10^{12}}{2 \text{ x } 200 \text{ x } 10^9} \text{ x } 10^{-4} \text{ x } 2 = 80\text{ J}$$

STRAIN ENERGY OF DEFORMATION:

2. Strain Energy due to Shear Stress:



Consider the a rectangular element subjected to shear as shown in fig. above. The height is ‘h’ and the plan area is ‘A’. It is distorted by a distance x due to shear force ‘F’. The graph of force plotted against ‘x’ is normally a straight line, so long as the material remains linearly elastic.

STRAIN ENERGY OF DEFORMATION:

Strain energy stored, $U = \frac{1}{2}Fx$

$$\text{Shear Stress}, \tau = \frac{F}{A} \qquad \text{Shear Strain}, \gamma = \frac{x}{h} \qquad x = \gamma.h$$

$$\text{Strain Energy}, U = \frac{1}{2}\tau.\gamma.A.h = \frac{1}{2}\tau.\gamma.\text{Volume}$$

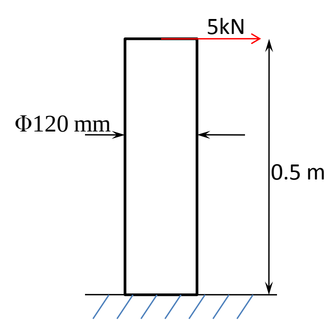
$$\text{Within the elastic limit}, \gamma = \frac{\tau}{G}$$

$$\text{Strain Energy}, U = \frac{1}{2}\frac{\tau^2}{G} \text{ x Volume of the block}$$

$$\text{In general strain energy}, U = \int_V \frac{1}{2}\frac{\tau^2}{G} dV$$

STRAIN ENERGY OF DEFORMATION:

Calculate the strain energy due to shear strain in the structure shown in fig $G = 90 \text{ GPa}$



$$\text{Area } A = \frac{\pi}{4} \text{ x } 120^2 \text{ x } 10^{-6} = 0.01131 \text{ m}^2$$

$$\text{Volume}, V = 0.01131 \text{ x } 0.5 = 5.6549 \text{ x } 10^{-3} \text{m}^2$$

$$\tau = \frac{F}{A} = \frac{5 \text{ x } 10^3}{0.01131} = 442.09 \text{ kN/m}^2$$

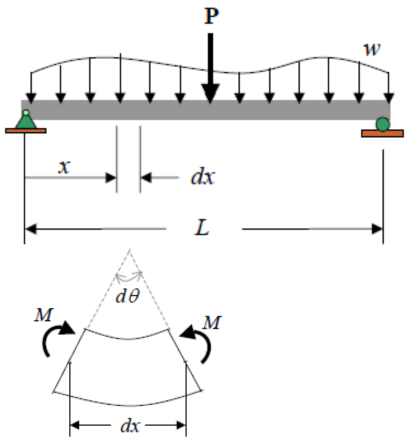
$$\text{Strain Energy}, U = \frac{1}{2}\frac{\tau^2}{G} \text{ x Volume}$$

$$= \frac{1}{2} \text{ x } \frac{(442.09 \text{ x } 10^3)^2}{90 \text{ x } 10^9} \text{ x } 5.6549 \text{ x } 10^{-3}$$

$$= 6.14 \text{ x } 10^{-3} \text{ Joules}$$

STRAIN ENERGY OF DEFORMATION:

3. Strain Energy due to Bending



$$\sigma = \frac{My}{I}$$

$$U_i = \int_V U_0 \, dV$$

$$= \int_V \left(\frac{1}{2}\sigma\varepsilon\right)(dV)$$

$$= \int_V \frac{1}{2}\left(\frac{\sigma^2}{E}\right) dV$$

$$= \int_V \frac{1}{2E}\left(\frac{My}{I}\right)^2 dV$$

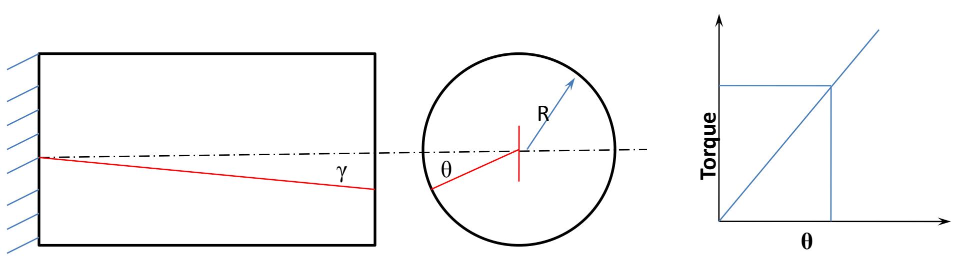
$$= \int_V \frac{1}{2E}\left(\frac{M^2 y^2}{I^2}\right) dV$$

$$= \int_L \frac{1}{2E}\left(\frac{M^2}{I^2}\right)\left(\int_A y^2 \, dA\right) dx \quad \rightarrow I$$

$$= \int_L \left(\frac{M^2}{2EI}\right) dx$$

STRAIN ENERGY OF DEFORMATION:

3. Strain Energy due to Torsion



The relation between torque T and the angle of twist θ is normally a straight line. Work done is the area under the torque angle graph. Strain Energy stored is given by $\mathbf{U = \frac{1}{2} \, . \, T . \, \theta}$

STRAIN ENERGY OF DEFORMATION:

$$\frac{T}{I_P} = \frac{\tau_{max}}{R} = \frac{G.\theta}{L}$$

I_P- Polar Moment of Inertia

R – Maximum radius of the shaft.

$$\theta = \frac{TL}{GI_P} \qquad T = \frac{\tau_{max}.I_P}{R}$$

$$\mathbf{U} = \frac{\mathbf{1}}{\mathbf{2}}.\frac{\tau_{max}.I_P}{R}.\frac{\tau_{max}.I_P}{R}.\frac{L}{G.I_P}$$

$$\mathbf{U} = \frac{\mathbf{1}}{\mathbf{2}}\frac{\tau_{max}{}^2.\pi.R^4}{R^2 \text{ x } 2}.\frac{L}{G} \qquad \mathbf{U} = \frac{\tau_{max}{}^2}{\mathbf{4G}} \textbf{ x Volume}$$

In general Strain Energy $\mathbf{U} = \int_V \frac{\tau_{max}}{\mathbf{4G}} \mathbf{dV}$

STRAIN ENERGY OF DEFORMATION:

A solid bar is 20 mm diameter and 0.8 m long. It is subjected to a torque of 30 N-m. Calculate the maximum shear stress and strain energy stored. Take G = 90 GPa.

$$\text{Max Shear Stress}, \tau_{max} = \frac{T.R}{I_P} = 19.1 \text{ x } 10^{-6}\text{Pa}$$

$$U = \frac{\tau^2}{4G} \text{ x Volume} = 0.255 \text{ Joules}$$

STRAIN ENERGY OF DEFORMATION:

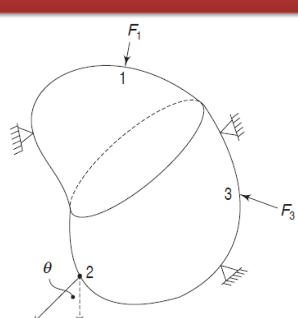
Strain Energy due to Axial Force $$\mathbf{U = \int_V \frac{\sigma^2}{2E} dV}$$

Strain Energy due to Shear Force $$U = \int_V \frac{1}{2}\frac{\tau^2}{G} dV$$

Strain Energy due to Torque $$U = \int_V \frac{\tau^2}{4G} dV$$

Strain Energy due to Bending Moment $$\mathbf{U = \int_0^l \frac{M^2}{2EI} dl}$$

STRAIN ENERGY OF DEFORMATION:



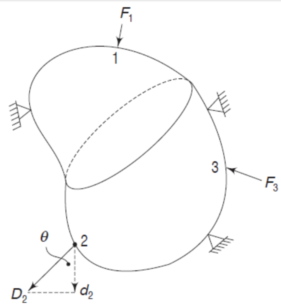
$$D_2 = k_{21} F_1$$

$$d_2 = D_2 \cos\theta = k_{21} \cos\theta \, F_1$$

$$d_2 = a_{21} F_1$$

Influence Coefficient: The displacement at point 2 in a specified direction due to a force F_1 applied at point 1 is proportional to F_1. The displacement produced at point 2 in a specified direction due to a unit force applied at point 1 is called **influence coefficient a_{21}**.

STRAIN ENERGY OF DEFORMATION:



Principle of Superposition : If several forces all having direction of F1 are applied simultaneously at 1 the resultant vertical deflection produced at 2 will be the resultant of deflection which they would have produced if applied separately. This is called **principle of superposition**.